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An oscillatory secondary bifurcation for magnetoconvection and rotating convection at small aspect ratio

By A. R. HALFORD AND M. R. E. PROCTOR†

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK

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We consider the dynamics of convection in a strong vertical magnetic field, and in the presence of rapid rotation. In both these cases, in circumstances which can be realized in the laboratory, the onset of convection is in the form of tall thin cells. Because of this, the dynamics near onset is characterized by an interaction between the cellular modes and the horizontally averaged temperature profile. The effects on the dynamics are slight in the case of a Boussinesq fluid. However in both cases, when the layer is stratified (non-Boussinesq), the convection can lose stability to oscillations close to onset. Properties of the oscillations and their stability to long-wavelength modulation are extensively investigated.

1. Introduction

There has been considerable interest in recent years in the development of reduced models of convection in fluid layers in the presence of rapid rotation about a vertical axis or strong vertical magnetic fields. It has been well known for many years (see e.g. Chandrashekhar 1961) that the onset of convection in these cases occurs on very small horizontal scales, and this permits a two-scale analysis of the problem. Proctor (1986) considered magnetoconvection in a Boussinesq fluid, and was able to write down a nonlinear evolution equation for the convection close to onset, in which the cellular motion interacted nonlinearly with the mean temperature profile. At leading order this system gave a strong selection of the wavenumber at onset, but little information as to the preferred planform, which had to be calculated at higher order. More recently Matthews (1999) and Julien, Knobloch & Tobias (1999, 2000) have extended the results in various ways, by looking at larger amplitudes and different scalings, and have produced many interesting results, particularly in determining the changes to the velocity profiles in the nonlinear regime, and non-trivial results for planform selection when the initial bifurcation is oscillatory. The rotating problem has recently been considered in the same spirit by Julien & Knobloch (1997, 1999) (see also Dawes 2001). Julien and his collaborators have considered aspects of the non-Boussinesq case. While they found the form of e.g. the vertical velocity far from onset, they did not investigate any effects of the asymmetry when the motion is on a hexagonal lattice.

The present paper therefore consists of an examination of the dynamics near the steady-state bifurcation on a hexagonal lattice in the non-Boussinesq case for

† Author to whom correspondence should be addressed.

both problems. It is known that for aspect ratios of order unity the dynamics is characterized by a standard normal form which guarantees steady solutions near onset, and shows that, for weakly non-Boussinesq conditions at least, there is stable steady convection with hexagonal planform near onset, provided that the parameters are such that there is no primary Hopf bifurcation. The new feature here, due to the small aspect ratio of the convection, is the presence of another dynamically active quantity, namely the horizontally averaged temperature perturbation. Remarkably, this new mode interacts with the terms in the normal form induced by the hexagonal symmetry so as to destabilize the steady hexagons close to onset, and to affect their stability to modes with less symmetry in a non-trivial way. It must be emphasized that the only special circumstance here is that the imposed magnetic field (or imposed rotation) is sufficiently large. The oscillations can occur for a large range of the diffusivity ratios, including (for the magnetic problem) the laboratory case where the magnetic diffusivity is very large compared with other diffusion rates. They have nothing to do with the oscillatory primary bifurcation that can occur only when the magnetic diffusivity is sufficiently small. Similarly, the oscillations in the rotating problem can occur for arbitrary sufficiently large Prandtl numbers. While the oscillations can be understood in the main by assuming them to be coherent in the horizontal directions, there is the possibility that they may lose coherence over long distances via mechanisms analogous to those driving the Turing and Benjamin–Feir instabilities.

The plan of the paper is as follows. In the next section we set up a simple model problem (magnetoconvection with variable magnetic diffusivity), and derive the governing equations for large Chandrasekhar number Q by expanding in powers of $Q^{-1/6}$. The resulting system consists of a p.d.e. in the horizontal (x, y) directions describing the cellular motion, and a p.d.e. in the vertical (z) direction describing the evolution of the mean temperature profile. This is then reduced to a set of o.d.e.s under further assumptions on the parameters. In § 3 we carry out the same task for a rapidly rotating layer with variable viscosity. This exhibits different scalings from the magnetic problem, but can be reduced to essentially the same model. In § 4 we investigate the dynamics of these o.d.e.s and also consider the effect of small terms representing degeneracy-breaking effects at higher order. Finally we model possible long-wavelength instabilities by introducing diffusion in the horizontal directions. In a conclusion we point the way to future work.

2. Magnetoconvection with non-uniform diffusivity ratio ζ

2.1. *Governing equations*

We consider thermal convection in a horizontal layer of depth d due to a temperature difference ΔT across the layer. In the basic motionless state a uniform vertical magnetic field $B_0\hat{z}$ permeates the layer. The gravity vector is $-g\hat{z}$. The fluid has constant thermal diffusivity κ and kinematic viscosity ν, while the magnetic diffusivity $\eta = \eta(z)$ is not constant. In all other respects the system obeys the Boussinesq approximation, in which densities etc. are taken to be constant except in the driving buoyancy term. It should be emphasized that the device of allowing η to depend on z is simply a convenient way of breaking the up–down symmetry that characterizes the Boussinesq approximation. More realistic representations of the effects of stratification may be expected to give qualitatively similar results. We non-dimensionalize distances with d (so that the layer lies between $z = 0$ and $z = 1$), the velocity $\boldsymbol{u}$ with κ/d, magnetic field perturbation $\boldsymbol{b}$ with B_0, and write the temperature $T = T_0 + \Delta T(1 - z + \theta)$, where T_0 gives the temperature at $z = 1$. Time t is scaled with d^2/κ, and fluid pressure p with $\kappa\nu/d^2$. We then obtain the following system (Proctor & Weiss 1982; Proctor 1986):

$$\frac{1}{\sigma}\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + R\theta\hat{\boldsymbol{z}} + Q\zeta_0\left(\frac{\partial \boldsymbol{b}}{\partial z} + \boldsymbol{b}\cdot\nabla\boldsymbol{b}\right) + \nabla^2\boldsymbol{u}, \tag{2.1}$$

$$\frac{\partial\theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta = \boldsymbol{u}\cdot\hat{\boldsymbol{z}} + \nabla^2\theta, \tag{2.2}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{b} = \frac{\partial \boldsymbol{u}}{\partial z} + \boldsymbol{b}\cdot\nabla\boldsymbol{u} - \zeta_0\nabla\times(h(z)\nabla\times\boldsymbol{b}), \tag{2.3}$$

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{b} = 0, \tag{2.4}$$

where $\sigma = \nu/\kappa$ is the Prandtl number, $\zeta_0 = \eta_0/\kappa$ (where $\eta_0 = \eta(0)$ is the diffusivity ratio at $z = 0$), $R = g\alpha\Delta T d^3/\kappa\nu$ is the Rayleigh number (α is the coefficient of thermal expansion) and $Q = B_0^2 d^2/(\mu_0\rho_0\nu\eta_0)$ is the Chandrasekhar number (μ_0 is the magnetic permeability and ρ_0 the base density). The important function $h(z) = \eta/\eta_0$ measures the variation of η with height. In Proctor (1986) $h(z)$ was set to unity. The boundary conditions at $z = 0, 1$ are the standard ones, namely $\theta = u_z = 0$, $\hat{\boldsymbol{z}}\times D\boldsymbol{u} = \hat{\boldsymbol{z}}\times\boldsymbol{b} = 0$, where D denotes a derivative with respect to z.

2.2. *Large-Q expansion*

Guided by earlier work, we now suppose that $Q \gg 1$; defining the small parameter $\epsilon \equiv Q^{-1/6}$, we make the following scalings:

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \epsilon^{-1}\left(\frac{\partial}{\partial \tilde{x}}, \frac{\partial}{\partial \tilde{y}}\right); \quad \boldsymbol{u} = (\epsilon\tilde{u}_x, \epsilon\tilde{u}_y, \tilde{u}_z) \equiv (\epsilon\tilde{\boldsymbol{u}}_H, \tilde{u}_z); \quad p = \epsilon^{-2}\tilde{p};$$

$$\theta = \epsilon^2(\tilde{\theta}(\tilde{x}, \tilde{y}, z) + \Theta(z)); \quad \boldsymbol{b} = (\epsilon^3\tilde{b}_x, \epsilon^3\tilde{b}_y, \epsilon^2\tilde{b}_z) \equiv (\epsilon^3\tilde{\boldsymbol{b}}_H, \epsilon^2\tilde{b}_z);$$

and write $R = \epsilon^{-6}r$. Θ represents the horizontally averaged part of the perturbation temperature, while $\tilde{\theta}$ now represents the fluctuating part and has zero mean by definition. These scalings follow those in Proctor (1986) but differ from those in other papers such as Matthews (1999), which deal with steady states much further from onset. (Note that in Proctor (1986) σ was also scaled to be of order ϵ^2. We could make this assumption here as well, but it adds little to the model.) We now drop the tildes in these expressions and expand all quantities in powers of ϵ^2, so that $\theta = \theta_0 + \epsilon^2\theta_2$, etc. Then at leading order we obtain

$$\mathscr{L}\begin{bmatrix} u_{0z} \\ \theta_0 \\ b_{0z} \end{bmatrix} \equiv \begin{bmatrix} r_0\theta_0 + \zeta_0 D b_{0z} \\ u_{0z} + \nabla_H^2\theta_0 \\ D u_{0z} + \zeta_0 h(z)\nabla_H^2 b_{0z} \end{bmatrix} = 0, \tag{2.5}$$

where ∇_H gives the horizontal component of the gradient. By eliminating all variables except θ_0, and restricting to solutions bounded in the horizontal, we make the *ansatz* $\theta_0 = -\nabla_H^2 F(x, y, t) f(z)$ where

$$D(h(z)^{-1}Df(z)) + r_0 f(z) = 0, \tag{2.6}$$

with the boundary conditions $f = 0$, $z = 0, 1$; this two-point boundary value problem determines r_0 as an eigenvalue. We fix F by imposing a normalization on f so that

$$\langle f^2\rangle \equiv \int_0^1 f^2\,\mathrm{d}z = 1. \tag{2.7}$$

The eigenfunction of $\mathscr{L}$ can then be written in the form

$$\begin{bmatrix} u_{0z} \\ \theta_0 \\ b_{0z} \end{bmatrix} = \begin{bmatrix} f\nabla_H^4 F \\ -f\nabla_H^2 F \\ -(\zeta_0 h)^{-1} Df\nabla_H^2 F \end{bmatrix}. \tag{2.8}$$

We next determine the horizontal components $\boldsymbol{u}_{0H}$, $\boldsymbol{b}_{0H}$. These may be found from the horizontal components of (2.1), (2.3) and from (2.4). We obtain

$$0 = -\nabla_H p_0 + \zeta_0 D\boldsymbol{b}_{0H}, \tag{2.9}$$

$$0 = D\boldsymbol{u}_{0H} + \zeta_0 h\nabla_H^2 \boldsymbol{b}_{0H} - \zeta_0 Dh\nabla_H b_{0z}, \tag{2.10}$$

$$0 = \nabla_H \cdot \boldsymbol{u}_{0H} + Du_{0z} = \nabla_H \cdot \boldsymbol{b}_{0H} + Db_{0z}. \tag{2.11}$$

These can be shown to be solved for $\boldsymbol{u}_{0H} = \Psi\nabla_H(-\nabla_H^2 F)$, $\boldsymbol{b}_{0H} = \Phi\nabla_H F$, where

$$\Psi = Df, \quad \Phi = \frac{1}{\zeta_0} D\left(\frac{1}{h} Df\right) = -\frac{r_0}{\zeta_0} f; \quad p_0 = \zeta_0 D\Phi F. \tag{2.12}$$

At the next order we obtain the evolution equation for F. This arises as a solvability condition for the inhomogeneous system

$$\mathscr{L}\begin{bmatrix} u_{2z} \\ \theta_2 \\ b_{2z} \end{bmatrix} = \begin{bmatrix} N^{(1)} \\ N^{(2)} \\ N^{(3)} \end{bmatrix}, \tag{2.13}$$

where

$$\left.\begin{aligned} N^{(1)} &= Dp_0 - r_2\theta_0 - \zeta_0(\boldsymbol{b}_{0H}\cdot\nabla_H b_{0z} + b_{0z}Db_{0z}) - \nabla_H^2 u_{0z}, \\ N^{(2)} &= \frac{\partial \theta_0}{\partial t} + \boldsymbol{u}_{0H}\cdot\nabla_H\theta_0 + u_{0z}D\theta_0 - D^2\theta_0, \\ N^{(3)} &= \frac{\partial b_{0z}}{\partial t} + \boldsymbol{u}_{0H}\cdot\nabla_H b_{0z} + u_{0z}Db_{0z} - \boldsymbol{b}_{0H}\cdot\nabla_H u_{0z} - b_{0z}Du_{0z} - \zeta_0 h(z)D^2 b_{0z}. \end{aligned}\right\} \tag{2.14}$$

By elimination among these equations we find that

$$r_0 u_{2z} + D\left(\frac{1}{h}Du_{2z}\right) = r_0 N^{(2)} + D\left(\frac{1}{h}N^{(3)}\right) - \nabla_H^2 N^{(1)}, \tag{2.15}$$

and so using (2.6) we have

$$\left\langle f\left(r_0 N^{(2)} + D\left(\frac{1}{h}N^{(3)}\right) - \nabla_H^2 N^{(1)}\right)\right\rangle = 0. \tag{2.16}$$

Application of this condition at every value of (x, y) gives the following equation for F:

$$\begin{aligned} (r_0 - \zeta_0^{-1}C_B^2)\nabla_H^2\left(\frac{\partial F}{\partial t}\right) &= r_0 H(t)\nabla_H^4 F - r_0 C_\theta^2\nabla_H^2 F + \nabla_H^8 F - r_2\nabla_H^4 F - C_N r_0\zeta_0^{-1} \\ &\quad\times[\nabla_H^4 F\nabla_H^2 F + \nabla_H F\cdot\nabla_H(\nabla_H^4 F) \\ &\quad + 2\nabla_H^2(\nabla_H^2 F)^2 - \nabla_H^2(\nabla_H F\cdot\nabla_H(\nabla_H^2 F))], \end{aligned} \tag{2.17}$$

where f has been normalized according to (2.7), and

$$C_B^2 = \left\langle \frac{1}{h^2}(Df)^2\right\rangle, \quad C_\theta^2 = \langle (Df)^2\rangle, \quad C_N = \left\langle \frac{1}{h} f^2 Df\right\rangle. \tag{2.18}$$

It is supposed that $r_0 > \zeta_0^{-1} C_B^2$, this being a sufficient condition for steady-state bifurcation from the zero velocity state. The function $H(t) = \langle f^2 D\Theta \rangle$, and the system is closed by the equation for Θ, obtained from the horizontal average of (2.2):

$$\frac{\partial \Theta}{\partial t} = 2(fDf)\,\overline{\nabla_H^4 F \nabla_H^2 F} + D^2\Theta. \tag{2.19}$$

When $h = 1$, we have $C_B^2 = C_\theta^2 = r_0 = \pi^2$, $C_N = 0$, and the results agree with equation (3.4) of Proctor (1986) (where $\nabla_H^2 F$ is written $-g$). However the character of the system is completely altered when h is not symmetric about the mid-plane, since then generically $C_N \neq 0$. For other non-symmetric problems such as convection in a polytropic atmosphere other quadratic terms may appear; for example in the present case there is no contribution from the nonlinear terms in $N^{(2)}$, but this cannot be expected to be true in general. The analysis of subsequent sections does not depend on the precise nature of the quadratic terms.

The equation for F given above resembles the 'long wavelength' equations for problems such as convection between poorly conducting boundaries or Marangoni convection, in which the horizontal scale is much greater than the vertical one (for references and examples see Knobloch 1990). Here the opposite is true; the key to the reduction in the present case is that the limiting critical Rayleigh number becomes independent of the horizontal scale as $Q \to \infty$.

2.3. *Reduction for small stratification*

Equations (2.17), (2.19) together present the curious aspect of a p.d.e. in (x, y, t) being coupled to one in (z, t), and their analysis, which has to be carried out numerically, is beyond the scope of the present paper. However we may reduce the system to four coupled o.d.e.s when $C_N \ll 1$ and the coefficient on the left-hand side of (2.17) is very small. To achieve this we suppose that h is almost equal to unity and $\zeta_0 \approx 1$, so that $C_N = \delta \tilde{C}$ and $r_0 - \zeta_0^{-1} C_B^2 = \delta^2 \gamma^2$, say where $\delta \ll 1$. (There are of course more general circumstances in which the reduction can be made.) If we make the scalings $F = \delta \tilde{F}$, $\Theta = \delta^2 \tilde{\Theta}$, $r_2 = r^{(0)} + \delta^2 r^{(2)}$, drop the tildes and expand quantities in powers of δ^2, so that $\tilde{F} = F^{(0)} + \delta^2 F^{(2)} + \ldots$, etc., then at leading order we obtain

$$0 = (-\pi^4 \nabla_H^2 + \nabla_H^8 - r^{(0)} \nabla_H^4) F^{(0)}. \tag{2.20}$$

We seek solutions periodic on a hexagonal lattice, so that

$$F^{(0)} = A(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}} + B(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}} + C(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}_3\cdot\boldsymbol{x}} + \text{c.c.}, \tag{2.21}$$

where $|\boldsymbol{k}_j| = k$, $j = 1, 2, 3$, and $\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 = 0$. Then $\nabla_H^2 F^{(0)} = -k^2 F^{(0)}$, and so $r^{(0)} = \pi^4 k^{-2} + k^4$. Because h is approximately unity, $f \approx \sqrt{2} \sin \pi z$ and $\tilde{\Theta} \approx -G(t) \sin 2\pi z$. Then (2.19) takes the form

$$\frac{\partial G}{\partial t} = 2\pi k^6 (|A|^2 + |B|^2 + |C|^2) - 4\pi^2 G, \tag{2.22}$$

while at $O(\delta^2)$ the solvability condition for the $F^{(2)}$ equation yields, when projected onto the three components of $F^{(0)}$,

$$\left.\begin{aligned} -\gamma^2 k^2 \frac{\partial A}{\partial t} &= k^4 \pi^3 G A - \tilde{r} k^4 A - \alpha B^* C^*, \\ -\gamma^2 k^2 \frac{\partial B}{\partial t} &= k^4 \pi^3 G B - \tilde{r} k^4 B - \alpha C^* A^*, \\ -\gamma^2 k^2 \frac{\partial C}{\partial t} &= k^4 \pi^3 G C - \tilde{r} k^4 C - \alpha A^* B^*, \end{aligned}\right\} \tag{2.23}$$

where $\tilde{r} = r^{(2)} - k^{-2}\delta^{-2}(r_0 C_\theta^2 - \pi^4)$, and $\alpha \propto \tilde{C}$. The asterisks denote complex conjugate. Finally we choose $k = k_c = (\pi^4/2)^{1/6}$ (to minimize $r^{(0)}$), and we rescale time, A, B, C and G to give the canonical system

$$\left.\begin{aligned} \dot{A} &= -GA + \mu A + B^* C^*, \\ \dot{B} &= -GB + \mu B + C^* A^*, \\ \dot{C} &= -GC + \mu C + A^* B^*, \\ \dot{G} &= s\left(\tfrac{1}{3}(|A|^2 + |B|^2 + |C|^2) - G\right), \end{aligned}\right\} \tag{2.24}$$

where $\mu \propto \tilde{r}$ and $s \propto \gamma^2/\alpha^2$. There is no rational reduction of the equations available when δ is not small. Nonetheless the strong selection of the wavenumber k_c at onset is likely to yield nonlinear solutions, periodic on a hexagonal lattice, for which the above equations give a good qualitative approximation. A full numerical investigation of (2.17), (2.19) is in progress.

3. Convection in a rapidly rotating layer with non-uniform viscosity

3.1. *Governing equations*

The dynamics of rapidly rotating convection has received as much attention as the magnetic problem; see Chandrasekhar (1961) for details and earlier references. The Boussinesq case at high amplitude has been treated by Julien & Knobloch (1997, 1999); see also Dawes (2001). Here, as in the magnetic problem, we focus on conditions near onset. We can perform a very similar analysis, leading to essentially the same canonical equations. We further suppose, again as a model of the effects of stratification, that the viscosity $\nu = \nu(z)$ and now define $h(z) = \nu(z)/\nu(0)$, $\nu(0) \equiv \nu_0$ by analogy with the h defined in the previous section. The vertical angular velocity of the layer is $\Omega\hat{z}$, and the Taylor number $T \equiv 4\Omega^2 d^4/\nu_0^2$ is taken to be large. The centrifugal force term is (as is conventional) supposed balanced by a horizontal pressure gradient. Making the same non-dimensionalization as before, we arrive at the system

$$\frac{1}{\sigma}\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + R\theta\hat{z} - T^{1/2}\hat{z}\times\boldsymbol{u} + \nabla\cdot(h\nabla\boldsymbol{u}), \tag{3.1}$$

$$\frac{\partial\theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta = \boldsymbol{u}\cdot\hat{z} + \nabla^2\theta, \tag{3.2}$$

$$\nabla\cdot\boldsymbol{u} = 0. \tag{3.3}$$

The boundary conditions on $\boldsymbol{u}$ and θ are the same as previously. The dimensionless numbers apart from T are the same as before, except that ν is replaced by ν_0.

3.2. *Large-T expansion*

For the Boussinesq problem (see e.g. Chandrasekhar 1961) it is found that in the limit of large T the critical Rayleigh number scales like $T^{2/3}$, while the critical wavenumber scales like $T^{1/6}$. We therefore define $\epsilon = T^{-1/6}$, and make the following scalings analogous to those in the previous section:

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \epsilon^{-1}\left(\frac{\partial}{\partial\tilde{x}}, \frac{\partial}{\partial\tilde{y}}\right) = \epsilon^{-1}\tilde{\nabla}_H; \quad p = \epsilon^{-2}\tilde{p}; \quad \theta = \epsilon^2(\tilde{\theta}(\tilde{x},\tilde{y},z) + \Theta(z));$$

$$\boldsymbol{u} = \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{v}}, \quad \tilde{\boldsymbol{u}} = (\epsilon\tilde{u}_x, \epsilon\tilde{u}_y, \tilde{u}_z) \equiv (\epsilon\tilde{\boldsymbol{u}}_H, \tilde{u}_z), \quad \tilde{\boldsymbol{v}} = (\tilde{v}_x, \tilde{v}_y, 0);$$

$\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{v}}$ may be identified with the poloidal and toroidal parts of the velocity field, and we define the respective potentials $\epsilon^2\tilde{\psi}$, $\epsilon\tilde{\phi}$ by $\tilde{\boldsymbol{u}} = (\epsilon\tilde{\nabla}_H(D\tilde{\psi}), -\tilde{\nabla}_H^2\tilde{\psi})$, $\tilde{\boldsymbol{v}} = \hat{z}\times\tilde{\nabla}_H\tilde{\phi}$.

Here the tilde indicates derivatives relative to the scaled coordinates as before. We also write $R = \epsilon^{-4} r$. We once again drop the tildes and expand all quantities in powers of ϵ^2. At leading order we obtain

$$\mathscr{L}\begin{bmatrix} p_0 \\ \psi_0 \\ \phi_0 \\ \theta_0 \end{bmatrix} \equiv \begin{bmatrix} -Dp_0 + r_0\theta_0 - h\nabla_H^4\psi_0 \\ \nabla_H(-p_0+\phi_0) \\ \hat{\boldsymbol{z}} \times \nabla_H(-D\psi_0 + h\nabla_H^2\phi_0) \\ -\nabla_H^2\psi_0 + \nabla_H^2\theta_0 \end{bmatrix} = 0. \tag{3.4}$$

The first three equations represent respectively the vertical poloidal, horizontal poloidal and toroidal parts of the momentum equation (3.1). If we write $\psi_0 = F(x,y,t)f(z)$ (note the change in the definition of F from the previous section), then in contrast to the magnetic problem F is not arbitrary but must satisfy the planform equation $\nabla_H^2 F = -k^2 F$. Using this, we can eliminate all variables in favour of f, which must satisfy the boundary value problem

$$D\left(\frac{1}{h}Df\right) + k^2(r_0 - k^4 h)f = 0; \quad f(0) = f(1) = 0. \tag{3.5}$$

The eigenvalue r_0 will depend on k, and we suppose that k has the value that minimizes r_0 (when $h=1$ we have $k = (\pi^2/2)^{1/6}$; $r_0 = 3k^4$). We normalize f according to equation (2.7). The other variables can then be written in terms of F, f:

$$\theta_0 = Ff; \quad \phi_0 = p_0 = -F\left(\frac{1}{k^2 h}\right)Df \equiv Fg(z). \tag{3.6}$$

At the next order we have an inhomogeneous system, namely

$$\mathscr{L}\begin{bmatrix} p_2 \\ \psi_2 \\ \phi_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} M^{(1)} \\ \nabla_H M^{(2)} \\ \hat{\boldsymbol{z}} \times \nabla_H M^{(3)} \\ M^{(4)} \end{bmatrix}, \tag{3.7}$$

where

$$\left.\begin{aligned} M^{(1)} &= \sigma^{-1}(\dot{u}_{0z} + \boldsymbol{u}_{0H}\cdot\nabla_H u_{0z} + u_{0z}Du_{0z}) - r_2\theta_0 - D(hDu_{0z}) \\ &= \sigma^{-1}(k^2\dot{F}f + (k^4F^2 + k^2|\nabla_H F|^2)fDf) - r_2Ff - k^2FD(hDf), \\ \nabla_H M^{(2)} &= \sigma^{-1}(\boldsymbol{v}_0\cdot\nabla_H\boldsymbol{v}_0) + k^2h\boldsymbol{u}_{0H} \\ &= \nabla_H\left(\sigma^{-1}(\tfrac{1}{2}g^2(k^2F^2 + |\nabla_H F|^2)) + hk^2FDf\right), \\ \hat{\boldsymbol{z}}\times\nabla_H M^{(3)} &= \sigma^{-1}(\dot{\boldsymbol{v}}_0 + \boldsymbol{v}_0\cdot\nabla_H\boldsymbol{u}_{0H} + \boldsymbol{u}_{0H}\cdot\nabla_H\boldsymbol{v}_0 + u_{0z}D\boldsymbol{v}_0) - D(hD\boldsymbol{v}_0) \\ &= \hat{\boldsymbol{z}}\times\nabla_H\left(\sigma^{-1}\left(Fg + \tfrac{1}{2}k^2F^2(fDg - gDf)\right) - FD(hDg)\right), \\ M^{(4)} &= \dot{\theta}_0 + \boldsymbol{u}_{0H}\cdot\nabla_H\theta_0 + u_{0z}D\theta_0 - D^2\theta_0 + u_{0z}D\Theta \\ &= \dot{F}f + (k^2F^2 + |\nabla_H F|^2)fDf - FD^2f + k^2FfD\Theta. \end{aligned}\right\} \tag{3.8}$$

The solvability condition for this system is (cf. equation (2.16))

$$\left\langle f\left(M^{(1)} - DM^{(2)} - \frac{1}{k^2}D\left(\frac{1}{h}M^{(3)}\right) + r_0\frac{1}{k^2}M^{(4)}\right)\right\rangle = 0. \tag{3.9}$$

To obtain the governing equations we must also project the nonlinear terms in F onto the space spanned by F. If we adopt the *ansatz* (2.21) then after some reduction we obtain

$$\left(\frac{k^2}{\sigma}+\frac{r_0}{k^2}-\frac{1}{\sigma}\langle g^2\rangle\right)\frac{\partial A}{\partial t}=A\left(r_2-2k^2\langle h(Df)^2\rangle-\frac{r_0}{k^2}\langle (Df)^2\rangle+\langle h(Dg)^2\rangle\right)$$
$$-Ak^2H(t)-\frac{k^2}{\sigma}B^*C^*\langle g^2Df\rangle, \quad (3.10)$$

together with two similar equations for B and C, where again $H(t)=\langle f^2 D\Theta\rangle$, and Θ obeys the analogue of (2.19), namely

$$\frac{\partial \Theta}{\partial t}=2(fDf)\,\overline{F\nabla_H^2 F}+D^2\Theta. \quad (3.11)$$

This system is already close to the canonical form (2.24) which was found for the magnetic problem, in that the equations for A, B, C are o.d.e.s. The p.d.e. for Θ can be formally reduced as in the previous section, by choosing $h(z)$ to be close to unity, and the coefficient of $\dot{A}$ in (3.10) to be small (this occurs for σ slightly greater than $1/3$). The reader may verify that in this case also the canonical form (2.24) emerges as the correct set of governing equations. This reduction is unfortunately of limited interest since for $\sigma<\sigma_0\approx 0.67$ the first bifurcation as R is increased is to oscillatory motion at a different wavenumber. Nonetheless, it is easily checked by simple numerical calculation that even when $\sigma\sigma_0$ the system (3.10), (3.11) with $A=B=C$ will have similar behaviour to (2.24), including the crucial Hopf bifurcation described in the next section. We therefore confine ourselves in what follows to the study of (2.24) and its extensions.

4. Analysis of the canonical system

4.1. *Solutions with hexagonal symmetry*

The equations (2.24) in principle describe solutions that have arbitrary initial amplitudes A, B, C. There are solutions in the form of rolls (e.g. $B=C=0$) when $\mu>0$; these evolve to the steady state $\frac{1}{3}|A|^2=G=\mu$ but it may be verified that this state is unstable to small perturbations. When all of A, B, C are non-zero we can show by writing $A=R_1\mathrm{e}^{\mathrm{i}\phi_1}$, etc. that there is evolution towards a steady state ($\phi_1+\phi_2+\phi_3=0$) in which all the variables may be taken as real and positive. In addition, as we show below, (see (4.11) and *ff*) the subspace where A, B, C have the same amplitude is linearly stable. We may therefore take each of these variables equal, real and positive. The system then reduces to the simple second-order system

$$\dot{A}=A(\mu+A-G), \quad \dot{G}=s(A^2-G). \quad (4.1)$$

The line $A=0$ is invariant (and A, being an amplitude, is restricted to being non-negative). The fixed point $\{O\,;(A,G)=(0,0)\}$ is stable for $\mu<0$. There is a steady-state bifurcation at $\mu=0$, and the non-trivial branch of steady modes satisfies

$$\mu+A_0-A_0^2=0, \quad G_0=A_0^2. \quad (4.2)$$

For $0>\mu>-\frac{1}{4}$ we have two solutions C_+, C_-, with $A_0>\frac{1}{2}$ for C_+ (see figure 1). The C_- solution is always unstable. There is clearly a saddle-node (SN) bifurcation at $\mu=-1/4$, but there is also the possibility of a Hopf bifurcation. Writing $A=A_0+a\mathrm{e}^{\lambda t}$, $G=G_0+g\mathrm{e}^{\lambda t}$ and linearizing in a, g we obtain the following equation for λ:

$$\lambda^2+\lambda(s-A_0)+s(2A_0^2-A_0)=0. \quad (4.3)$$

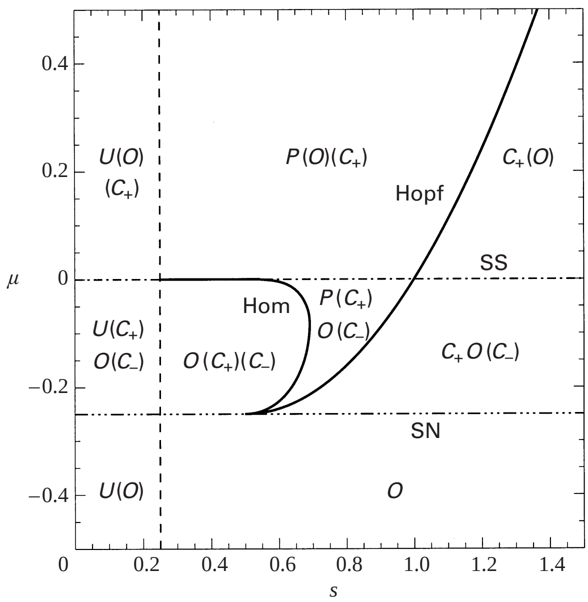

FIGURE 1. Bifurcation diagram for the system (4.1). Possible solution types are: O, origin; $C_\pm$, steady convection; P, periodic oscillation. Unstable solutions are shown in brackets. The symbol U indicates that some solutions can escape to infinity.

Thus the C_+ state is stable only for $A_0 < s$. If $s > \frac{1}{2}$ there is a Hopf bifurcation at $A_0 = s$, $\mu = s(s-1)$. If $s < \frac{1}{2}$ the whole non-trivial steady branch is unstable. The point $(\mu, s) = (-\frac{1}{4}, \frac{1}{2})$ is a standard Takens–Bogdanov double-zero bifurcation point (see e.g. Chapter 7 of Guckenheimer & Holmes 1986) and we can use the general theory given there to show that in the neighbourhood of this point (for $s > \frac{1}{2}$) there will be a supercritical Hopf bifurcation from C_+ with the resulting periodic solution becoming homoclinic to C_-. Further away from the degeneracy we have investigated the equation numerically. We find that the Hopf bifurcation is always supercritical. If we decrease s at fixed μ we obtain two different behaviours according to whether μ is positive or negative. For $-\frac{1}{4} < \mu < 0$ the steady solution C_+ loses stability on the line labelled 'Hopf' in figure 1. The periodic orbit formed in the bifurcation grows until it becomes homoclinic to the solution C_- on the line 'Hom'. Thereafter the solution tends to O for generic initial conditions until $s = \frac{1}{4}$. To the left of this line the solution can escape to infinity for sufficiently large initial conditions (denoted by U in figure 1). We discuss the large-amplitude dynamics below. For $\mu > 0$, on the other hand, finite-amplitude homoclinic orbits are impossible since the only available saddle point, namely O, lies in an invariant plane. In this case stable periodic orbits exist all the way down to $s = \frac{1}{4}$. There is a finite attracting region as long as $s > \frac{1}{4}$, but otherwise solutions can be unbounded whatever the value of μ. The various regions in parameter space are shown in figure 1, while examples of periodic orbits are shown in figure 2.

We can investigate the behaviour of the solution when $\mu > 0$ when the solutions are of large amplitude. This will occur for the periodic orbits when s is close to $\frac{1}{4}$ or when μ is large. The orbits are then characterized by a 'fast phase', in which A starts small, makes a large excursion and returns to low amplitude, and a 'slow phase' in which A is very small. A similar type of analysis was used for a different convection

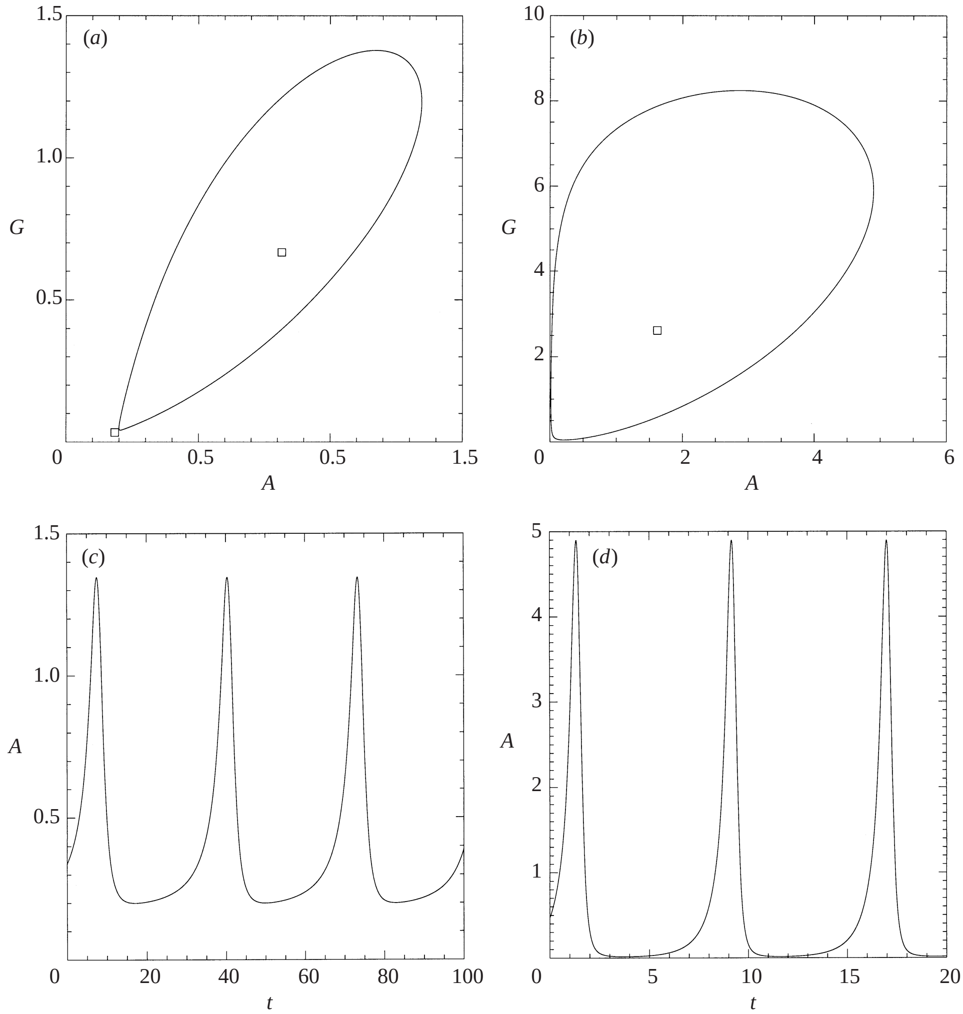

FIGURE 2. Two examples of limit cycle solutions of (4.1). In (*a*) and (*c*) $s = 0.68$ and $\mu = -0.15$, while in (*b*) and (*d*), $s = 1.0$ and $\mu = 1.0$. The squares show the locations of the fixed points.

problem by Hughes & Proctor (1990*a, b*). In the slow phase we have approximately

$$\dot{A} = (\mu - G)A, \quad \dot{G} = -sG, \tag{4.4}$$

and so

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}G} = \frac{1}{s}\left(1 - \frac{\mu}{G}\right). \tag{4.5}$$

Integrating this equation between $(A, G) = (\varDelta, G_1)$ and $(\varDelta, G_2)$, where the (small) value $\varDelta$ labels the value of A at the beginning and end of the slow phase, we find

$$G_1 - \mu \log G_1 = G_2 - \mu \log G_2. \tag{4.6}$$

In the fast phase, on the other hand, we may use the approximation

$$\dot{A} = A(\mu + A - G), \quad \dot{G} = sA^2, \tag{4.7}$$

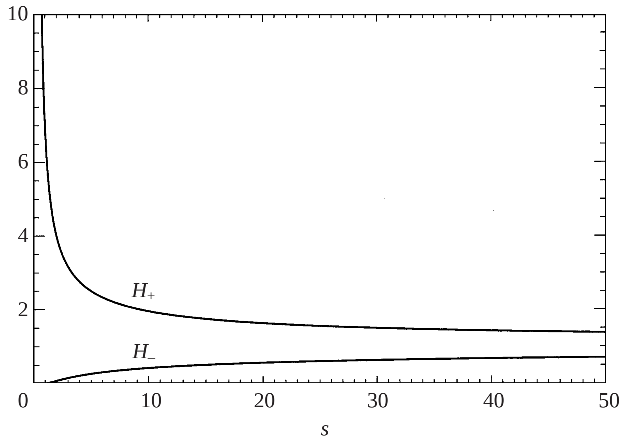

FIGURE 3. Fixed points H_+, H_- of the maps (4.6), (4.9) (with μ set to unity), as functions of s (the actual fixed points are found by multiplying by μ). There are no solutions for $s < \frac{1}{4}$.

and since A is never zero, we may define a stretched time variable τ by $\mathrm{d}/\mathrm{d}t = A(\mathrm{d}/\mathrm{d}\tau)$. Then the equations are linear, and we have the solution, satisfying $A \approx 0$ and $G \approx G_2$ at $\tau = 0$,

$$\left.\begin{aligned} A &= \frac{1}{q}(\mu - G_2)\mathrm{e}^{p\tau}\sin q\tau, \\ G &= \mu + (G_2 - \mu)\mathrm{e}^{p\tau}\left(\cos q\tau - \frac{1}{q}(1-p)\sin q\tau\right), \end{aligned}\right\} \tag{4.8}$$

where $p \pm \mathrm{i}q$ are the roots of the characteristic equation $\lambda^2 - \lambda + s = 0$, so that $p = \frac{1}{2}$, $q = \frac{1}{2}\sqrt{4s-1}$. The fast phase comes to an end when A becomes small again, or when $\tau = \pi/q$. This occurs for $G = G_3$, where

$$G_3 - \mu = (\mu - G_2)\Gamma, \quad \Gamma = \mathrm{e}^{p\pi/q}. \tag{4.9}$$

For consistency we must have $G_3 \gg 1$, and so it is necessary that $\mu \gg 1$ or else $\Gamma \gg 1$ ($q \ll 1$). The pair of mappings (4.6), (4.9) can be shown to have stable fixed points $G_1 = G_3 = \mu H_+(s)$, $G_2 = \mu H_-(s)$ for all $s > \frac{1}{4}$; the functions H_+, H_- are shown in figure 3.

When $s < \frac{1}{4}$ the characteristic equation above has real positive roots, and it may then be shown that solutions of sufficiently large amplitude will typically escape to infinity.

4.2. *Higher-order corrections in the magnetic case*

The reduced system (4.1) does not contain any higher-order terms that discriminate between rolls and hexagons; it is possible that in the full system the hexagons seen near onset lose stability to rolls at larger amplitude, as they do when Q is not large. If we are to understand the behaviour of the convection at larger amplitudes, we should add in such degeneracy-breaking terms, which are neglected in the leading-order analysis. Rather than carry out an involved calculation, we can represent the effect of neglected terms of higher order by adding arbitrary small terms to the equations for A, B, C. We are thus led to consider the system

$$\left.\begin{aligned}\dot{A} &= -GA + \mu A + B^*C^* - A(\eta_1|A|^2 - \eta_2(|B|^2 + |C|^2)),\\ \dot{B} &= -GB + \mu B + C^*A^* - B(\eta_1|B|^2 - \eta_2(|C|^2 + |A|^2)),\\ \dot{C} &= -GC + \mu C + A^*B^* - C(\eta_1|C|^2 - \eta_2(|A|^2 + |B|^2)),\\ \dot{G} &= s\left(\tfrac{1}{3}(|A|^2 + |B|^2 + |C|^2) - G\right),\end{aligned}\right\} \qquad (4.10)$$

which retains the symmetries of the underlying convection problem. Strictly speaking we should add terms to the G equation also, proportional to e.g. G^2 or $ABC + A^*B^*C^*$. It will be seen, however, that these have no effect on the symmetry-breaking criteria, since in (4.11) and similar equations the dynamics of G decouples. We take A, B, C to be real as before and first look at the hexagonal case $A = B = C$. Then the non-trivial steady branch satisfies $\mu + A - (1 + \tilde{\eta})A^2 = 0$, where $\tilde{\eta} = \eta_1 + 2\eta_2$. We shall suppose this quantity to be positive, guaranteeing bounded solutions. It can then be easily shown that there is no Hopf bifurcation for $s > 1/8\tilde{\eta}$ while for $1/(2(1 + \tilde{\eta})^2) < s < 1/8\tilde{\eta}$ there are two bifurcation points, with just one for $s < 1/(2(1 + \tilde{\eta})^2)$. In every case the steady solution is stable for sufficiently large μ.

We now examine perturbations that lead to a breaking of the hexagonal symmetry. Consider a state (possibly time-dependent) that is stable in the symmetric subspace, for which $A = B = C = D$, $G = H$ say. Then writing $A = D(1 + a), \ldots, G = H + g$ and linearizing, we obtain

$$\left.\begin{aligned}\dot{a} &= -g - a(D + 2\eta_1 D^2) + (b + c)(D - 2\eta_2 D^2),\\ &\vdots\\ \dot{g} &= s(-g + 2D^2(a + b + c)),\end{aligned}\right\} \qquad (4.11)$$

and so the equation for $\Lambda = a - b$ decouples from that for g and takes the simple form $\dot{\Lambda} = \Lambda(-2D + 2(\eta_2 - \eta_1)D^2)$. There is thus a symmetry-breaking bifurcation (to rectangles), when $\langle D \rangle = (\eta_2 - \eta_1)\langle D^2 \rangle$, where the brackets now indicate an average over a period of D. Hexagons remain stable if $\eta_2 < \eta_1$. From the equations satisfied by the symmetric state, and recalling that $D > 0$, we have $\langle D^2 \rangle = \langle H \rangle = \mu + \langle D \rangle - \tilde{\eta}\langle D^2 \rangle$. Rearranging, we obtain the criterion for marginal stability,

$$\mu = \frac{1 + 2\eta_1 + \eta_2}{\eta_2 - \eta_1}\langle D \rangle. \qquad (4.12)$$

This result indicates that μ has to be very large for the hexagons to be unstable. But in fact for small $\tilde{\eta}$ the average of D can be much smaller than the corresponding value at the fixed point. For example, if $\tilde{\eta} = 0.01$, $\mu = 20$, $s = 0.325$ the value of D at the fixed point is 4.97, but the average of D is only 0.453. Thus the value of μ at which the solution with hexagonal symmetry becomes unstable is reduced by more than a factor of ten, and this ratio does not change very much with μ, and so it seems that the stability boundary occurs at values of μ rather smaller than of order $\tilde{\eta}^{-1}$ as implied by (4.12).

For $\mu > 0$ we also have steady solutions of 'roll' type, with e.g. $A > 0$, $B = C = 0$. These satisfy $\mu = A^2(\frac{1}{3} + \eta_1)$. The stability problem for perturbations to B and C is decoupled from G, and so the results are as in the usual case ($s \to \infty$), namely that rolls are stable for $\eta_2 > \eta_1$, $\mu > (\eta_2 - \eta_1)^{-2}(\frac{1}{3} + \eta_1)$. This is about one third the value for marginal stability for hexagons, derivable from (4.12) when D represents the fixed point. However there is the possibility of a Hopf bifurcation within the roll subspace, when $\eta_1 < 0$, and in this case the rolls are unstable when $\mu > s(1 + \eta_1)/(-6\eta_1)$. Except for very large values of s this occurs at much lower values of μ than the rectangular instability, so that there is generally no region where rolls are stable if $\eta_1 < 0$. The bifurcation is in fact degenerate at leading order, and amplitudes tend to infinity in the unstable region if yet-higher-order terms are neglected. Thus for the eventual stability of rolls we should generally require $\eta_1 > 0$.

Because of the reduction of the value of μ for rectangular instability due to the oscillations we have the possibility that the (time-periodic) hexagons lose stability while the steady rolls are still unstable. This would suggest that it might be possible to obtain stable time-periodic 'rectangles', for which $A = B \neq C$. Such solutions can be found with considerable difficulty, but always in the region for which the rolls are unstable to the oscillatory instability described above. They are thus not of great interest.

4.3. *Higher-order corrections in the rotating case*

The rotating case differs from the magnetic case only in that the former has broken reflectional symmetry, while the latter does not. We will therefore obtain reduced equations precisely analogous to (4.10), which differ from those equations only by replacing terms like $\ldots \eta_2(|B|^2 + |C|^2)$ by $\ldots (\eta_2|B|^2 + \eta_3|C|^2)$, *mutatis mutandis* cyclically in the other equations. The difference between η_2 and η_3 is due to the lack of reflectional symmetry, which is not broken in the magnetic field problem.

The instabilities of the last subsection have their counterpart here, with $2\eta_2$ being replaced by $\eta_2 + \eta_3$. The results on stability within the hexagonal subspace are unchanged. For the rectangular instability, we define (compare the definition of Λ above) $\Lambda_1 = (a - b)$, $\Lambda_2 = (b - c)$, and obtain the equations

$$\left.\begin{aligned} \dot{\Lambda}_1 &= \Lambda_1(-2D + 2(\eta_3 - \eta_1)D^2) + 2(\eta_3 - \eta_2)D^2\Lambda_2, \\ \dot{\Lambda}_2 &= \Lambda_2(-2D + 2(\eta_2 - \eta_1)D^2) + 2(\eta_2 - \eta_3)D^2\Lambda_1. \end{aligned}\right\} \tag{4.13}$$

Using the new variables $Q_1 = \sqrt{3}(\Lambda_1 + \Lambda_2)$, $Q_2 = \Lambda_1 - \Lambda_2$ we find

$$\left.\begin{aligned} \dot{Q}_1 &= Q_1(-2D + (\eta_2 + \eta_3 - 2\eta_1)D^2) + \sqrt{3}(\eta_3 - \eta_2)D^2Q_2, \\ \dot{Q}_2 &= Q_2(-2D + (\eta_2 + \eta_3 - 2\eta_1)D^2) - \sqrt{3}(\eta_3 - \eta_2)D^2Q_1, \end{aligned}\right\} \tag{4.14}$$

and so it follows that we have instability of hexagons when $(\eta_2+\eta_3-2\eta_1)\langle D^2\rangle > 2\langle D\rangle$. (Steady) rolls are stable if $\mu(\eta_1 - \eta_2)(\eta_1 - \eta_3) > (\frac{1}{3} + \eta_1)$. When one of the factors on the left-hand side is negative, then rolls are always unstable and we have the analogue of the Kuppers–Lortz instability (Busse & Heikes 1980). This leads to a periodic modulation of the roll amplitudes; the structurally stable cycle that arises in the Boussinesq case does not appear here (see Swift 1984 and Soward 1985).

We have estimates of η_2 and η_3 thanks to the pioneering analysis of Soward (1974), who looked (in the course of a derivation of a convection-driven dynamo model) at the Boussinesq problem. He considered an intermediate scaling between the one used here and the strongly nonlinear limit of Julien and collaborators; in his scaling $r - r_0 = \mathcal{O}(\epsilon)$ and the energy $|A|^2 + |B|^2 + |C|^2$ is constant, so (necessarily) $\eta_1 = \eta_2 + \eta_3 = 0$ at leading order. In this case rolls are always unstable and hexagons neutrally stable. When small non-Boussinesq effects are included, hexagons are stabilized for all $\langle D\rangle$. To investigate the eventual destabilization of hexagons at larger amplitudes requires delicate higher-order analysis.

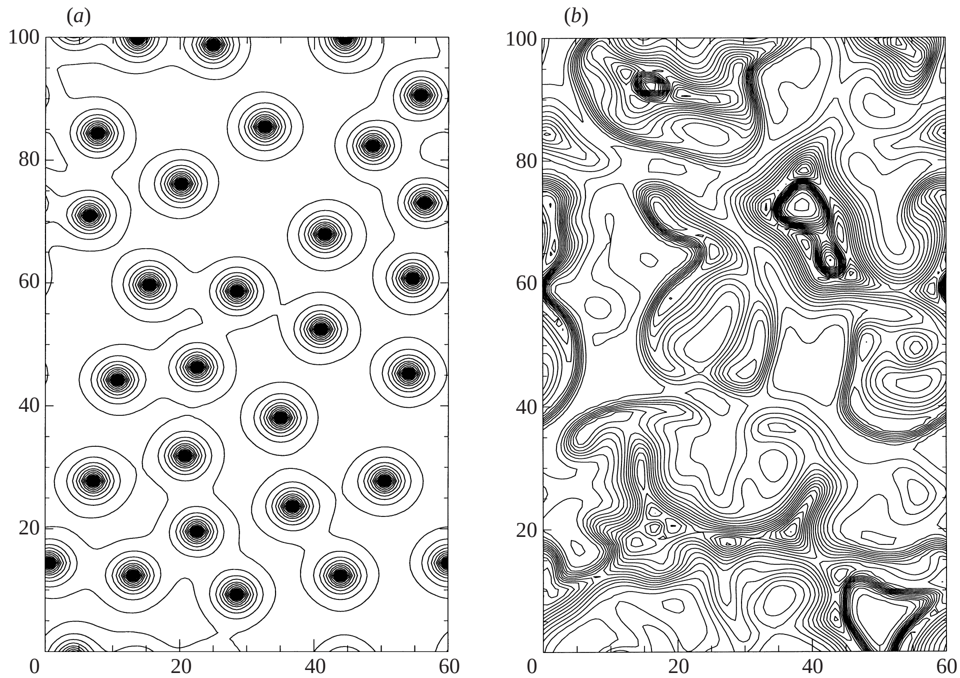

FIGURE 4. Contours of G in equation (5.1). In (a) $s = 2$, $\mu = -35/144$, $\kappa_1 = 1$, $\kappa_2 = 16$ (Turing mode); this is an (almost) steady state. In (b) $s = 1$, $\mu = 1$, $\kappa_1 = 1$, $\kappa_2 = 1/9$ (Benjamin–Feir mode). This is a snapshot of a state disordered in space and time.

5. Modulational instabilities

We now return to the basic equations (4.1) and consider the effects of large-scale spatial dependence. We suppose the hexagons to be modulated on long horizontal scales (X, Y). Rather than carry out a higher-order analysis of the equations, we model the effect of modulation by adding isotropic diffusion terms to the right-hand sides of the equations. If we were to proceed with full generality, we would allow three different modulated roll amplitudes, by appropriately extending (2.24). In fact the mechanisms which stabilize the hexagonal planform in the absence of modulation still operate here, with the consequence that A, B, C are attracted to a subspace where they are equal in amplitude and can be taken as real. In that case the symmetries of the pattern make the isotropic Laplacian operator the natural one. Then we have the following equations, where now A, G are functions of X, Y, t and $\kappa_{1,2}$ are positive diffusion coefficients:

$$\dot{A} = A(\mu + A - G) + \kappa_1 \nabla^2 A, \quad \dot{G} = s(A^2 - G) + \kappa_2 \nabla^2 G. \tag{5.1}$$

This is of a form similar to reaction–diffusion equations encountered in chemistry and biology. There exists the possibility of Turing-type instability. If we examine the linearized equation for small disturbances to a steady solution (A_0, A_0^2), and replace ∇^2 by $-k^2$, we obtain the dispersion relation analogous to (4.3), namely

$$\lambda^2 + \lambda(s + (\kappa_1 + \kappa_2)k^2 - A_0) + (s + \kappa_2 k^2)(\kappa_1 k^2 - A_0) + 2sA_0^2 = 0. \tag{5.2}$$

Apart from the Hopf bifurcation at $k = 0$, $A_0 = s$ there are Turing-type modes that arise when the constant term in (5.2) is negative. This can occur when $A_0\kappa_2 > s\kappa_1$ provided $(A_0\kappa_2 - s\kappa_1)^2 > 4\kappa_1\kappa_2(2sA_0^2 - sA_0)$. These conditions can be satisfied when

$\kappa_2 > 2s\kappa_1 > \kappa_1$, since $s > \frac{1}{2}$ for there to be a uniform steady state stable to Hopf modes. In that case there is a Turing instability when $\frac{1}{2} < A_0 < s(2\sqrt{2s}\gamma - \gamma^2)^{-1}$, where $\gamma = \sqrt{\kappa_2/\kappa_1}$. The mode appears and disappears before the appearance of the Hopf mode in the range $\sqrt{2s} < \gamma < \sqrt{2s} + \sqrt{2s-1}$ and in this case we find, in the unstable range, a steady pattern of modulation with a wavenumber of order unity. If on the other hand $2\sqrt{2s} > \gamma > \sqrt{2s} + \sqrt{2s-1}$, the Hopf bifurcation occurs in the Turing unstable range, and we expect that the oscillations that arise will be Turing unstable too. The lack of symmetry under sign change in the governing equations implies that the pattern of modulation will be hexagonal close enough to onset, though further away an irregular pattern of 'cells' can be observed with random initial conditions (see figure 4).

A completely different long-wave instability mode can occur when the spatially uniform state is periodic in time. This instability, distinguished by zero critical wavenumber, is analogous to the Benjamin–Feir instability of water waves (see e.g. Stuart & Di Prima 1978). It depends on the fact that a periodic solution of an autonomous system is always neutrally stable to a mode representing a simple time shift. Let us suppose that (4.1) has a time-periodic solution $A_0(t), G_0(t)$. The linear stability of this state to long-wavelength modulations can be found by defining a small parameter ϵ and adopting the *ansatz* $A = A_0(\tau) + \epsilon^2 r_a$; $G = G_0(\tau) + \epsilon^2 r_g$; $\tau = t + \phi(\tilde{X}, \tilde{Y}, T)$ where $T = \epsilon^2 t$, $(\tilde{X}, \tilde{Y}) = \epsilon(X, Y)$ are slow time and length scales, and (r_a, r_g) are remainder terms. It is easy to see that correct to $O(\epsilon^2)$, $\dot{A} = \dot{A}_0(1 + \epsilon^2\phi_T) + \epsilon^2\dot{r}_a, \nabla^2 A = \epsilon^2(\dot{A}_0\tilde{\nabla}^2\phi + \ddot{A}_0|\tilde{\nabla}\phi|^2)$, and similarly for G. Then substituting into the p.d.e.s and dropping the tildes we find at $O(\epsilon^2)$

$$\left.\begin{aligned} \dot{r}_a - r_a(\mu + 2A_0 - G_0) + A_0 r_g &= (\kappa_1\nabla^2\phi - \phi_T)\dot{A}_0 + \kappa_1|\nabla\phi|^2\ddot{A}_0, \\ \dot{r}_g - 2sA_0 r_a + s r_g &= (\kappa_2\nabla^2\phi - \phi_T)\dot{G}_0 + \kappa_2|\nabla\phi|^2\ddot{G}_0. \end{aligned}\right\} \tag{5.3}$$

To eliminate secular growth on the fast time scale, we apply a solvability condition to (5.3). We define $(\rho_a(t), \rho_g(t))$ to be functions with the same period as A_0 satisfying the adjoint problem

$$\dot{\rho}_a = -\rho_a(\mu + 2A_0 - G_0) - 2sA_0\rho_g, \quad \dot{\rho}_g = A_0\rho_a + s\rho_g. \tag{5.4}$$

It is then easy to show that

$$\phi_T = K\nabla^2\phi + L|\nabla\phi|^2, \tag{5.5}$$

where

$$K = \frac{\kappa_1\langle\dot{A}_0\rho_a\rangle + \kappa_2\langle\dot{G}_0\rho_g\rangle}{\langle\dot{A}_0\rho_a\rangle + \langle\dot{G}_0\rho_g\rangle}, \quad L = \frac{\kappa_1\langle\ddot{A}_0\rho_a\rangle + \kappa_2\langle\ddot{G}_0\rho_g\rangle}{\langle\ddot{A}_0\rho_a\rangle + \langle\ddot{G}_0\rho_g\rangle}, \tag{5.6}$$

and the angle brackets now denote an average over a period of A_0. We can linearize this equation by the substitution $\psi = \mathrm{e}^{L\phi/K}$, giving $\psi_T = K\nabla^2\psi$. For simplicity we define $\mathscr{W} = -\langle\dot{A}_0\rho_a\rangle/\langle\dot{G}_0\rho_g\rangle$, then we find that K can be negative, indicating modulational instability to infinitesimal disturbances, if either $\mathscr{W} < 1$, $\kappa_1\mathscr{W} > \kappa_2$ or $\mathscr{W} > 1, \kappa_1\mathscr{W} < \kappa_2$.

We have sought values for $\mathscr{W}$ by obtaining numerical solutions to (5.4). Because of the sign of the time derivatives, this has to be done by finding a solution for $A_0(t)$, $G_0(t)$ and integrating backwards in time over several periods to achieve convergence. In every case we looked at we found that $0 < \mathscr{W} < 1$ and so the patterns are unstable to the Benjamin–Feir mode for sufficiently small κ_2/κ_1. (L is found to be positive.) These conclusions have been tested by carrying out numerical simulations of (5.1) in a periodic domain of size $60 \times 60\sqrt{3}$. Figure 4 shows modulated patterns for (*a*) $s = 2$, $\mu = -35/144$, $\kappa_1 = 1$, $\kappa_2 = 16$ (Turing mode), and (*b*) $s = 1$, $\mu = 1$, $\kappa_1 = 1$, $\kappa_2 = 1/9$ (Benjamin–Feir mode).

Of course, the coefficients will depend on the physical problem and the parameters. But from the above it would seem that there is the potential for modulation over almost the entire range of κ_1/κ_2.

6. Discussion

This paper, by means of two simple examples, has demonstrated that when the tendency to hexagonal convection due to the breaking of up–down symmetry is combined with the dynamical inclusion of the mean temperature mode due to the small aspect ratio at high magnetic field strengths (or high Taylor number in the rotating case), new and interesting time-dependent behaviour can result not far from onset. The most novel and unexpected aspect of the results is the appearance of a Hopf bifurcation leading to oscillatory motion of the hexagonal pattern. While the full dynamics is described by a complicated set of differential equations, it is possible to make a reduction that allows extensive investigation of the time-dependence. We are also able to describe the effects of the new instability on the breakdown of the hexagonal symmetry. Finally, we show that the uniform hexagonal pattern can be expected to break down due to modulational instabilities. While the details of the analysis involve several special choices of parameters, the principal governing equations (2.17) and (3.10), together with the mean temperature equations, hold for a wide range of parameters. In particular, there is no restriction imposed on the diffusivity ratio ζ_0 or the Prandtl number σ, except that they must be large enough that the primary bifurcation is steady-state. Similarly the stratification is not restricted to be small, except for carrying out the full reduction in the magnetic case. The only important requirement is that the imposed magnetic field (or rotation rate for the second problem) must be sufficiently large (or equivalently, the aspect ratio of the convection at onset sufficiently small) that the secondary instabilities described will occur before any other kind of breakdown of the steady solution. Thus these transitions can in principle be observed in the laboratory, and are not mere abstractions of interest only to the theorist.

It is interesting to note that the nonlinear terms in the amplitude equations appear at dominant order in the scaling used, which follows that of Proctor (1986). In the scaling of Matthews (1999) and Julien *et al.* (1999, 2000) all linear fields appear at one order lower in ϵ. Thus in their problem the symmetry-breaking quadratic terms in (2.17) would appear at higher order in the expansion than the term involving the mean temperature, and it might therefore be thought that their results would remain valid even when up–down symmetry is broken. This would seem to be the case for steady solutions; consideration of the steady versions of (2.17), (2.19) shows that in the limit of large r_2 the quadratic terms are a small perturbation. However we know that the instability described here does not vanish at large r_2 when the degeneracy is not broken. It is thus a nice question when and if these larger-amplitude solutions regain stability as the Rayleigh number increases beyond the range of our theory. We have provided a partial answer to this by showing that other small degeneracy-breaking terms can restabilize the steady state (see discussion below (4.10)) but this does not happen in a region where we can trust the expansion.

Of course the analysis presented here for the reduced system, while demonstrating the existence of oscillatory convection in one parameter regime, does not tell the full story. We are in the process of integrating the full equations (2.17), (2.19) in order to determine precisely the region of instability.

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